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Generalized coherent states and spin $S \geqslant 1$ systems

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Abstract. Generalized Coherent States (GCS) are constructed (and discussed) in order to study quasiclassical behaviour of quantum spin models of the Heisenberg type. Several such models are taken to their semiclassical limits, whose form depends on the spin value as well as the Hamiltonian symmetry. In the continuum approximation, $SU(2)/U(1)$ GCS when applied give rise to the well known Landau–Lifshitz classical phenomenology. For arbitrary spin values one obtains a lattice of coupled nonlinear oscillators. Corresponding classical continuum models are described as well.

1. Introduction

In what follows our main aim is to provide a semiclassical description of quantum spin models of Heisenberg type. For the sake of simplicity we confine ourselves to considering Heisenberg ferromagnets (and antiferromagnets) with uniaxial anisotropy of the following form:

$$\hat{H}_{\mathrm{e}} = \mp J \sum_n (\hat{\boldsymbol{S}}_n \hat{\boldsymbol{S}}_{n+1} + \delta \hat{S}_n^z \hat{S}_{n+1}^z) \tag{1}$$

for exchange anisotropy, or

$$\hat{H}_{\mathrm{s}} = \mp J \sum_n (\hat{\boldsymbol{S}}_n \hat{\boldsymbol{S}}_{n+1} + \delta \hat{S}_n^z \hat{S}_n^z) \tag{2}$$

for single-ion anisotropy. Here $\hat{S}_n^x$, $\hat{S}_n^y$, $\hat{S}_n^z$ are the spin operators acting at a site n, and δ is the anisotropy coefficient.

In the one-dimensional case with spin $s = \frac{1}{2}$, the study can be carried out completely [1]. Higher-spin (and higher-dimensional) models require an approximate treatment. One of them we use later is the so-called trial function method. Based on minimization of the Hamiltonian with respect to a set of trial functions this method is in fact very sensitive to the choice of these functions. There are certain thoughts (ideas) and even theorems aiding in the search for them. Usually these ideas are based on symmetry properties of the system under consideration (e.g., the Coleman–Palais theorem). Symmetry turns out to be the most powerful and effective tool here. This is why we choose generalized coherent states (GCS) as the trial functions for models (1) and (2). Up to now there is an extensive literature devoted to the different aspects of the theory of coherent states and its applications. We only give here references to several original papers and a couple of books where a reader

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can find more detailed bibliography [2–4, 7]. However, in order to give a consistent picture of the problem, later on we briefly describe features of the GCS in question.

The paper is organized as follows. In section 2, we construct GCS defined on the homogeneous (complex projective) spaces $SU(2j+1)/SU(2j)\otimes U(1) \equiv \mathbb{CP}^{2j}$. The emphasis will be on the $SU(2)/U(1)$ GCS, reducing the quantum description to classical Landau–Lifshitz phenomenology. In the other cases, we shall only be interested in GCS in the faithful (fundamental) representation of corresponding groups. Also GCS constructed on the non-compact group $SU(1,1)$ will be presented in order to treat the antiferromagnet models (+ sign in (1) and (2)).

In section 3, classical counterparts of the spin operators and their products are obtained and discussed.

Section 4 is devoted to deriving classical lattice Hamiltonians for various spin values, s, starting from equations (1) and (2). Here, we note that the dimensions of the spin phase space at a lattice site coincides with that of the corresponding coset space. Indeed, we have for an arbitrary spin state at a site

$$|\Psi\rangle = \sum_{m=1}^{2s+1} c_m |\psi_m\rangle \tag{3}$$

where $|\psi_m\rangle$ are pure spin states, e.g., $|\psi_m\rangle = |s, m\rangle$, and c_m are complex constants. The vector $|\Psi\rangle$ is defined up to an arbitrary phase:$|\Psi\rangle \simeq |\Psi\rangle e^{i\theta}$ and should satisfy the normalization condition

$$\langle\Psi|\Psi\rangle = 1.$$

These two real conditions reduce the dimension of the spin phase space, $\mathbb{S}$, by two:

$$\dim \mathbb{S} = 2(2s+1) - 2 = 4s. \tag{4}$$

The dimension of $\mathbb{CP}^{2s}$ is

$$\dim \mathbb{CP}^{2s} = (2s+1)^2 - 4s^2 - 1 = 4s \tag{5}$$

so

$$\dim \mathbb{CP}^{2s} = \dim \mathbb{S}$$

and $\mathbb{CP}^{2s}$ GCS provide a complete description of the corresponding model.

Section 5 deals with classical lattice equations of motion derived by considering the quantum probability amplitude.

Section 6 considers continuum models resulting from the classical lattice models. Some properties of these continuum models are discussed.

2. GCS defined on the coset space $SU(2j+1)/SU(2j)\otimes U(1)$

Generalized coherent states (GCS), the extension of the well known Glauber coherent states related to the Heisenberg-Weyl group, are defined and discussed in detail in [2, 3]. Here we chose them as the trial functions for at least three reasons:

(i) Our aim is a semiclassical description of spin quantum models and GCS usually give such a description since they minimize the uncertainty relation.

(ii) They provide a minimal description (no extra parameters) as we'll see later on.

(iii) They are taken according to the demands of Hamiltonian symmetry.

Therefore our concern in what follows will be with GCS defined on the complex projective spaces, $\mathbb{CP}^{2j}$. The simplest is the sphere, $\mathbb{S}^2 = \mathbb{CP}^1 = SU(2)/U(1)$. To its points are related the so called spin coherent states discussed in detail in many papers and books (see, e.g., [2, 3]). These states can be parametrized by the points of a real sphere, $\mathbb{S}^2$ or, via the stereographic projection onto the complex plane, $\mathbb{C}$, by its points:

$$|\Psi\rangle = \mathrm{e}^{\alpha\hat{S}^+ - \bar{\alpha}\hat{S}^-}|0\rangle = (1+|\psi|^2)^{-j}\mathrm{e}^{\psi\hat{S}^+}|0\rangle \tag{6}$$

with

$$\hat{S}^\pm = \hat{S}^x \pm \mathrm{i}\hat{S}^y \qquad \text{and} \qquad \psi = \frac{\alpha}{|\alpha|}\tan|\alpha|$$

where α and ψ are complex numbers; $|0\rangle = |j, -j\rangle$, the ground state and j defines the unitary representation of the group, $SU(2)$, and hence the spin value: $s = j$. By stereographic projection, we have $\psi = -\tan\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}$, $|\alpha| \leqslant \frac{1}{2}\pi$. Thus, the set of GCS (6) has spherical symmetry and is (in principle) valid for any spin value. It means that the system (6) can be used as the trial function basis when the symmetry of the quantum Hamiltonian is very close to spherical symmetry, i.e. $\delta = 0$ or $\delta \ll 1$.

GCS for other groups are constructed using their fundamental representation

$$\begin{aligned}|\Psi\rangle &= \exp\left\{\sum_i^{2s}(\zeta_i\hat{T}_i^+ - \bar{\zeta}_i\hat{T}_i^-)\right\}|0\rangle \\ &= \left(1+\sum_i^{2s}|\psi_i|^2\right)^{-\frac{1}{2}}\left\{|0\rangle + \sum_i^{2s}\psi_i|i\rangle\right\}\end{aligned} \tag{7}$$

where $\hat{T}_i^+$ and $\hat{T}_i^-$ are generators of the $SU(2j+1)$ group in the fundamental representation and

$$\psi_i = \frac{\zeta_i}{|\zeta|}\tan|\zeta| \qquad |\zeta| = \sqrt{\sum_i^{2s}|\zeta_i|^2}$$

$$|0\rangle = (0,\ldots,0,1)^{\mathrm{tr}} \qquad |i\rangle = (0,\ldots,0,1,\underbrace{0,\ldots,0}_{i})^{\mathrm{tr}}.$$

For example, in the $\mathbb{CP}^2$ case we have $G = SU(3)$, the coset space $G/H = SU(3)/SU(2)\otimes U(1)$ and

$$|\Psi\rangle = (1+|\psi_1|^2+|\psi_2|^2)^{-\frac{1}{2}}\{|0\rangle + \psi_1|1\rangle + \psi_2|2\rangle\} \tag{8}$$

Let us consider GCS for $s = 1$ systems. In this case, states (6) are

$$|\Psi\rangle = \frac{1}{1+|\psi|^2}\{|0\rangle + \sqrt{2}\psi|1\rangle + \psi^2|2\rangle\}. \tag{9}$$

These states are parametrized by a complex function, ψ, i.e. this system lives on a two-dimensional real manifold (the spin phase space). The states (8) are parametrized by two complex functions, ψ_1 and ψ_2, so the system lives on a four-dimensional real manifold (the spin phase space). We saw earlier that the spin phase space in the $s = 1$ case was four-dimensional and the second system in this regard was more appropriate.

One can easily check that under the conditions

$$\psi_1 = \sqrt{2}\psi \qquad \psi_2 = \tfrac{1}{2}\psi_1^2$$

both systems coincide. It means that the first 2D manifold is just a section of the second 4D manifold, under the constraint

$$\psi_2 = \tfrac{1}{2}\psi_1^2. \tag{10}$$

An analogous projection will occur for $s = \frac{3}{2}$ where

$$|\Psi\rangle = (1 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2)^{-\frac{1}{2}} \big(|0\rangle + \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle\big) \tag{11}$$

and the constraint determining the two-dimensional $SU(2)$ section is

$$\psi_2 = \frac{1}{\sqrt{3}}\psi_1^2 \qquad \psi_3 = \frac{1}{3\sqrt{3}}\psi_1^3 \,. \tag{12}$$

Here it is useful to stress that, in the case $s = 1$, the states (8) can be rewritten in the so-called angle parametrization (see, for example, [4]):

$$|\Psi\rangle = \mathrm{e}^{-\mathrm{i}\phi \hat{S}^z} \mathrm{e}^{-\mathrm{i}\theta \hat{S}^y} \mathrm{e}^{\mathrm{i}\gamma \hat{S}^z} \mathrm{e}^{2ig\hat{Q}^{xy}} |0\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \tag{13}$$

where

$$\begin{aligned}
\hat{Q}^{xy} &= \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
c_0 &= \mathrm{e}^{\mathrm{i}\phi}(\mathrm{e}^{-\mathrm{i}\gamma} \sin^2 \tfrac{\theta}{2} \cos g + \mathrm{e}^{\mathrm{i}\gamma} \cos^2 \tfrac{\theta}{2} \sin g) \\
c_1 &= \frac{\sin\theta}{\sqrt{2}} (\mathrm{e}^{-\mathrm{i}\gamma} \cos g + \mathrm{e}^{\mathrm{i}\gamma} \sin g) \\
c_2 &= \mathrm{e}^{-\mathrm{i}\phi}(\mathrm{e}^{-\mathrm{i}\gamma} \cos^2 \tfrac{\theta}{2} \cos g + \mathrm{e}^{\mathrm{i}\gamma} \sin^2 \tfrac{\theta}{2} \sin g) \,.
\end{aligned} \tag{14}$$

The two angles, θ and ϕ, define the orientation of the classic spin vector. The angle, γ, is the rotation of the quadrupole moment about the spin vector. The parameter, g, defines change of the spin vector magnitude and that of the quadrupole moment.

Finally, we give GCS defined on the non-compact manifold $SU(1,1)/U(1)$ the two sheet hyperboloid $\mathbb{S}^{1,1}$:

$$|\zeta\rangle = (1 - |\zeta|^2)^k \mathrm{e}^{\zeta \hat{K}^+} |0\rangle \tag{15}$$

where again ζ is a complex number, $\hat{K}^+ = \hat{K}^x + \mathrm{i}\hat{K}^y$, and $|0\rangle$ is the ground state: $\hat{K}^-|0\rangle = 0$, i.e. $|0\rangle = |k, k\rangle$.

3. Averaged spin operators and their products

Here we consider classical counterparts of the spin operators and their products contained in the Hamiltonians (1) and (2).

The vector

$$\boldsymbol{S} = \langle\Psi|\hat{\boldsymbol{S}}|\Psi\rangle \tag{16}$$

can be regarded as a classical spin vector, and

$$Q^{ij} = \langle\Psi|\hat{S}^i \hat{S}^j|\Psi\rangle \tag{17}$$

as a component of the quadrupole moment.

Because the spin operators at different lattice sites commute, for all such products we have

$$\langle\Psi|\hat{S}_n^i \hat{S}_{n+1}^k|\Psi\rangle = \langle\Psi|\hat{S}_n^i|\Psi\rangle\langle\Psi|\hat{S}_{n+1}^k|\Psi\rangle \tag{18}$$

where

$$|\Psi\rangle = |\Psi\rangle_n |\Psi\rangle_{n+1}.$$

The corresponding expressions for the $\mathbb{CP}^1$ GCS are well known [2, 3]:

$$S^+ = \bar{S}^- = 2j\frac{\bar{\psi}}{1+|\psi|^2} \qquad S^z = -j\frac{1-|\psi|^2}{1+|\psi|^2} \tag{19}$$

$$Q^{zz} = \langle\Psi|\hat{S}^z\hat{S}^z|\Psi\rangle = \frac{j^2(1-|\psi|^2)+2j|\psi|^2}{(1+|\psi|^2)^2}$$

$$= (S^z)^2\left[1+\frac{2}{j}\frac{|\psi|^2}{(1-|\psi|^2)^2}\right] \equiv (S^z)^2 + \frac{1}{2j}S^+S^- . \tag{20}$$

The quasiclassical limit requires

$$2j \gg \frac{S^+S^-}{(S^z)^2} .$$

For the same reason for

$$Q^{+-} = S^+S^-\left(1+\frac{1}{2j}\frac{j+S^z}{j-S^z}\right) \qquad \text{or} \qquad Q^{-+} = S^+S^-\left(1+\frac{1}{2j}\frac{j-S^z}{j+S^z}\right) \tag{21}$$

the quasiclassical limits mean

$$2j \gg \frac{j+S^z}{j-S^z} \qquad \text{or} \qquad 2j \gg \frac{j-S^z}{j+S^z} .$$

One can also see that any quadrupole moment component, Q^{ij}, is expressed, though nonlinearly, through two components of the spin vector $\boldsymbol{S}$. Higher-order moments, Q^{ijk}, and so on are readily shown to be expressed, in a similar way.

The components of the classical spin vector, $\boldsymbol{S}$, and of the quadrupole moment, Q^{ij}, for other sets of GCS are less known, though straightforward:

$$S^+ = \bar{S}^- = \sqrt{2}\frac{\bar{\psi}_1+\psi_1\bar{\psi}^2}{1+|\psi_1|^2+|\psi_2|^2} \qquad (j=1) \tag{22}$$

$$S^z = \frac{|\psi_2|^2-1}{1+|\psi_1|^2+|\psi_2|^2} \qquad (j=1) \tag{23}$$

$$Q^{zz} = \frac{1+|\psi_2|^2}{1+|\psi_1|^2+|\psi_2|^2} \qquad (j=1) \tag{24}$$

(all other components of Q^{ij} can be expressed in terms of S^+, S^z and Q^{zz}).

$$S^+ = \bar{S}^- = \frac{\sqrt{3}\psi_2\bar{\psi}_3+2\psi_1\bar{\psi}_2+\sqrt{3}\bar{\psi}_1}{1+|\psi_1|^2+|\psi_2|^2+|\psi_3|^2} \qquad (j=\tfrac{3}{2}) \tag{25}$$

$$S^z = \frac{3|\psi_3|^2+|\psi_2|^2-|\psi_1|^2-3}{1+|\psi_1|^2+|\psi_2|^2+|\psi_3|^2} \qquad (j=\tfrac{3}{2}) \tag{26}$$

$$Q^{zz} = \frac{1}{4}\frac{9+|\psi_1|^2+|\psi_2|^2+9|\psi_3|^2}{1+|\psi_1|^2+|\psi_2|^2+|\psi_3|^2} \qquad (j=\tfrac{3}{2}) . \tag{27}$$

We note here that the averaged Casimir operators are

$$\hat{\mathbb{C}}_2 = \tfrac{1}{2}(Q^{+-}+Q^{-+})+Q^{zz} = j(j+1) \qquad \text{for the } \mathbb{CP}^1 \text{ GCS}$$

$$\hat{\mathbb{C}}_2 = j(j+1) = 2 \qquad \text{for the } \mathbb{CP}^2 \text{ GCS}$$

$$\hat{\mathbb{C}}_2 = j(j+1) = \tfrac{15}{4} \qquad \text{for the } \mathbb{CP}^3 \text{ GCS.}$$

More important is to note that for the $\mathbb{CP}^1$ GCS

$$\boldsymbol{S}^2 = \text{constant} = j^2. \tag{28}$$

For the $\mathbb{CP}^2$ GCS

$$\boldsymbol{S}^2 + q^2 = 1 \qquad (j = 1) \tag{29}$$

with

$$q^2 = Q^{z+}Q^{z-} + Q^{+z}Q^{-z} + Q^{++}Q^{--} + (1 - Q^{zz})^2 . \tag{30}$$

The analogous formula for the $\mathbb{CP}^3$ GCS is

$$\boldsymbol{S}^2 + q^2 + t^2 = \text{constant}$$

with more cumbersome expressions for q^2 involving Q^{ij} and t^2 through Q^{ijk}.

We emphasize that, in the $\mathbb{CP}^1$ case $(j = 1)$ by use of the angle parametrization (14), one has [8]

$$\begin{aligned} S^+ &= \mathrm{e}^{\mathrm{i}\phi} \cos 2g \sin\theta \\ S^z &= \cos 2g \cos\theta \\ q^2 &= \sin^2 2g \;\; \textit{and} \\ Q^{zz} &= 1 - \tfrac{1}{2}\sin^2\theta + \tfrac{1}{2}\sin^2\theta \cos 2\gamma \sin 2g . \end{aligned} \tag{31}$$

The identity $S^2 + q^2 = 1$ is trivially satisfied.

Finally, we give the expressions for the operators $\hat{K}^{\pm}$ and $\hat{K}^z$ (the generators of the group $SU(1, 1)$) averaged over the corresponding $\mathbb{L}^{1,1} = SU(1, 1)/U(1)$ GCS:

$$K^+ = \bar{K}^- = \langle|\hat{K}^+|\rangle = 2k \frac{\bar{\zeta}}{1 - |\zeta|^2} \tag{32}$$

$$K^z = k \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \tag{33}$$

$$K^{zz} = \langle|\hat{K}^z\hat{K}^z|\rangle = (K^z)^2 \left(1 + \frac{2}{k}\frac{|\zeta|^2}{(1 + |\zeta|^2)^2}\right) = (K^z)^2 + \frac{1}{2k} K^+K^- \tag{34}$$

$$K^{+-} = K^+K^- \left(1 + \frac{1}{2k}|\zeta|^2\right) \qquad K^{-+} = K^+K^- \left(1 + \frac{1}{2k}|\zeta|^{-2}\right) \tag{35}$$

such that the averaged Casimir operator is

$$\mathbb{C}_2 = \tfrac{1}{2}(K^{+-} + K^{-+}) - K^{zz} = k(1 - k)$$

and the classical pseudovector $\boldsymbol{K} = (K^x, K^y, K^z)$ obeys the condition

$$K^+K^- - (K^z)^2 = -k^2 \tag{36}$$

and lies on the two-sheet hyperboloid $\mathbb{S}^{1,1}$ (we omit the details which appear in [2]).

4. Classical lattice Hamiltonians

In this part, we derive classical lattice Hamiltonians which are just the Hamiltonians (1) and (2) averaged over different GCS.

(i) First we use the spin coherent states and consider the model (1). As has already been mentioned the spin operators at neighbouring sites commute, so the coherent state of the whole lattice is

$$|\Psi\rangle = \prod_n |\Psi\rangle_n \,. \tag{37}$$

Averaging (1) with (37) and using equations (19) in the ferromagnetic case one has

$$\begin{aligned} H_{\mathrm{e}} = \langle\Psi|\hat{H}_{\mathrm{e}}|\Psi\rangle &= -J\sum_n (\boldsymbol{S}_n\boldsymbol{S}_{n+1} + \delta S_n^z S_{n+1}^z) \\ &= -j^2 J \sum_n \frac{2(\bar{\psi}_n\psi_{n+1} + \mathrm{CC}) + (1+\delta)(1-|\psi_n|^2)(1-|\psi_{n+1}|^2)}{(1+|\psi_n|^2)(1+|\psi_{n+1}|^2)} \end{aligned} \tag{38}$$

the classical lattice Hamiltonian which in the continuum limit ($d = 1$) becomes

$$H_{\mathrm{e}} = -j^2 JN + J\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{a_0}\left(\frac{a_0^2}{2}\boldsymbol{S}_x\boldsymbol{S}_x - \delta S^z S^z\right) \tag{39}$$

for the σ-model representation, or

$$H_{\mathrm{e}} = -\mathrm{constant} + 2j^2 J a_0 \int_{-\infty}^{\infty} \mathrm{d}x\; \frac{|\psi_x|^2 + \rho|\psi|^2}{(1+|\psi|^2)^2} \tag{40}$$

for the stereographic projection with $\delta = \frac{1}{2}a_0^2\rho$. N is the total number of lattice sites.

From equation (40), we see that the classical energy above the ground state (a large negative constant) is positive and that the classical excitations will possess positive energy. In contrast, in the antiferromagnetic case direct application of the spin GCS leads to excitations with negative energy, i.e. such a system should be unstable (more correctly, the quantum vacuum over which we construct the excitations is unstable). Apparently, this is the reason why the search for the vacuum in the antiferromagnetic case is such a complicated problem. To avoid this difficulty and provide excitations with positive energy, following [2, 6] we use the following trick. Rewrite (1) via operators of the $su(1,1)$ algebra

$$\hat{K}^{\pm} = \mathrm{i}\hat{S}^{\pm} \qquad \hat{K}^z = \hat{S}^z. \tag{41}$$

Then we have the pseudospin representation for antiferromagnet:

$$\hat{H}_{\mathrm{e}} = -J\sum_n \left[\tfrac{1}{2}(\hat{K}_n^+\hat{K}_{n+1}^- + \mathrm{HC}) - \hat{K}_n^z\hat{K}_{n+1}^z(1+\delta)\right]. \tag{42}$$

We now treat this model applying the above scheme and using $\mathbb{L}^{1,1}$ GCS to obtain the classical lattice model:

$$H_{\mathrm{e}} = -k^2 J \sum_n \frac{2(\bar{\zeta}_n\zeta_{n+1} + \zeta_n\bar{\zeta}_{n+1}) - (1+\delta)(1+|\zeta_n|^2)(1+|\zeta_{n+1}|^2)}{(1-|\zeta_n|^2)(1-|\zeta_{n+1}|^2)}. \tag{43}$$

The continuum limits are

$$\begin{aligned} H_{\mathrm{e}} &= -J\sum_n \boldsymbol{K}^2 + \frac{a_0}{2}J\int \mathrm{d}x\;(\boldsymbol{K}_x\boldsymbol{K}_x + \rho K^z K^z) \\ &= Jk^2 N + \frac{a_0}{2}\int \mathrm{d}x\;(\boldsymbol{K}_x\boldsymbol{K}_x + \rho K^z K^z) \end{aligned} \tag{44}$$

for the σ-model representation, or

$$H_{\rm e} = \text{constant} + 2k^2 a_0 J \int \mathrm{d}x \; \frac{|\zeta_x|^2 + \rho|\zeta|^2}{(1-|\zeta|^2)^2} \tag{45}$$

for the stereographic projection.

We thus avoid the problem of excitations with negative energy, but are instead faced with the problem of treating non-compact groups and manifolds (σ-model representation) or singular expressions (stereographic projection).

The models (38)–(40) and (43)–(45) can be regarded as appropriate if the anisotropy constant δ is very small: $\delta \ll 1$. Then, the symmetries of the quantum Hamiltonians and of GCS manifolds should be very close: spherical for the ferromagnet (1) and pseudospherical for antiferromagnet (1). While taking to the continuum limit, we require $a_0/\lambda \ll 1$ where λ is the wavelength considered.

The same procedure can be applied to models (2) to give

$$\begin{aligned} H_{\rm s} &= \langle\Psi|\hat{H}|\Psi\rangle = -J\sum_n (\boldsymbol{S}_n\boldsymbol{S}_{n+1} + \delta Q_n^{zz}) \\ &= -J\sum_n \left\{\boldsymbol{S}_n\boldsymbol{S}_{n+1} + \delta\left[(S_n^z)^2 + \frac{1}{2j}S_n^+ S_n^-\right]\right\} \end{aligned} \tag{46}$$

or

$$H_{\rm s} = -j^2 J \sum_n \left[\frac{2(\bar{\psi}_n\psi_{n+1} + \mathrm{CC}) + (1-|\psi_n|^2)(1-|\psi_{n+1}|^2)}{(1+|\psi_n|^2)(1+|\psi_{n+1}|^2)} + \delta\frac{(1-|\psi_n|^2)^2 + \frac{2}{j}|\psi_n|^2}{(1+|\psi_n|^2)^2}\right]. \tag{47}$$

In the continuum limit

$$H_{\rm s} = -j^2 JN\left(1+\frac{\delta}{2j}\right) + \frac{a_0}{2}J\int \mathrm{d}x\;\left[\boldsymbol{S}_x\boldsymbol{S}_x - \delta\left(1-\frac{1}{2j}\right)(S^z)^2\right] \tag{48}$$

for the σ-model representation, or

$$H_{\rm s} = -j^2 JN\left(1+\frac{\delta}{2j}\right) + 2a_0 j^2 J\int \mathrm{d}x\;\frac{|\psi_x|^2 + \rho_1|\psi|^2}{(1+|\psi|^2)^2} \qquad \rho_1 = \rho\left(1-\frac{1}{2j}\right) \tag{49}$$

for the stereographic projection.

These formulae imply both systems being equivalent, up to renormalization of the following constants: the ground state energy level and the anisotropy rate. Hence, the classical dynamics of both systems is the same (with the exception of the $j = \frac{1}{2}$ case).

If we are given the Hamiltonian (2) with a $-$ sign, then

$$H_{\rm s} = k^2 JN\left(1-\frac{\delta}{2k}\right) + \frac{a_0}{2}J\int \mathrm{d}x\;\left[\boldsymbol{K}_x\boldsymbol{K}_x + \delta\left(1+\frac{1}{2k}\right)(K_z)^2\right] \tag{50}$$

for the σ-model representation, or

$$H_{\rm s} = k^2 JN\left(1-\frac{\delta}{2k}\right) + 2a_0 k^2 J\int \mathrm{d}x\;\frac{|\zeta_x|^2 + \rho_2|\zeta|^2}{91-|\zeta|^2)^2} \qquad \rho_2 = \rho\left(1+\frac{1}{2k}\right) \tag{51}$$

for the stereographic projection. Again both systems are equivalent.

Let us now proceed to other GCS.

(ii) Here we use $\mathbb{CP}^2$ GCS for treating the model (1) with $s = 1$. Then we have the same expressions for the Hamiltonians in the σ-model representation, namely equations (38), (39) and (44). In the complex representation, we have the lattice Hamiltonian

$$H_{\mathrm{e}} = -j^2 J \sum_n \left[\frac{(\bar{\psi}_{1n} + \psi_{1n}\bar{\psi}_{2n})(\psi_{1n+1} + \bar{\psi}_{1n+1}\psi_{2n+1}) + \mathrm{CC}}{(1+|\psi_{1n}|^2+|\psi_{2n}|^2)(1+|\psi_{1n+1}|^2+|\psi_{2n+1}|^2)} \right.$$
$$\left. + \frac{(1+\delta)(1-|\psi_{2n}|^2)(1-|\psi_{2n+1}|^2)}{(1+|\psi_{1n}|^2+|\psi_{2n}|^2)(1+|\psi_{1n+1}|^2+|\psi_{2n+1}|^2)} \right]. \tag{52}$$

The corresponding continuum expression is very complicated and we do not give it here for the general case. But to understand the difference between the two models, we must study their *ground states*. It is natural to suggest that, at least in the ferromagnetic case, this state will be close to the one for which

$$\psi_n = \psi_l \qquad \text{for} \quad n \neq l. \tag{53}$$

Then, the Hamiltonian of model (38) is as follows

$$H_{\mathrm{e}} = -J \sum_n (\boldsymbol{S}_n^2 + \delta (S_n^z)^2) = -j^2 JN - \delta J \sum_n (S_n^z)^2 \tag{54}$$

and the energy assumes its minimum when
(a) $\delta > 0$ (easy-axis model)

$$\boldsymbol{S} = (0, 0, S^z) \equiv (0, 0, j) \qquad E = -j^2 JN(1+\delta). \tag{55}$$

(b) $\delta < 0$ (easy-plane model)

$$\boldsymbol{S} = (S^x, S^y, 0) \qquad E = -j^2 JN. \tag{56}$$

For the $\mathbb{CP}^2$ model, we have (due to $s^2 + q^2 = 1$ at $j = 1$)

$$H_{\mathrm{e}} = -J \sum_m (\boldsymbol{S}_n^2 + \delta (S_n^z)^2) = -JN - J \sum_n (\delta (S_n^z)^2 - q_n^2) \tag{57}$$

with q_n^2 given by (30).

The ground states again are the states with $q_n^2 = 0$. However, in looking for excitations in the frame of the $\mathbb{CP}^1$ model, the term

$$H_{1e} = -J \sum_n \delta (S_n^z)^2 \tag{58}$$

is varied. In the $\mathbb{CP}^2$ case

$$H_{2e} = -J \sum_n [\delta (S_n^z)^2 - q_n^2] \tag{59}$$

i.e. here an additional term, $\sum_n q_n^2$, proportional to quadrupole moment and of effective anisotropy in nature, appears. Moreover in the first case classical dynamics is orientation dynamics such that the classical spin vector being of constant value just alters its direction (lies on the sphere $\mathbb{S}^2$). In the second case the vector can alter its value (along with quadrupole moment) as well as the direction.

The $\mathbb{CP}^3$ model, besides the quadrupole moment, q, has an octupole moment, t, and, hence, one more term:

$$H_{3e} = -J \sum_n [\delta (S_n^z)^2 - q_n^2 - t_n^2] \tag{60}$$

and so on.

Consider model (2). Here in the $\mathbb{CP}^1$ case, we have the Hamiltonian (58) with $\delta_1 = \delta(1 - 1/2j)$. But for the $\mathbb{CP}^2$ model

$$H_{\mathrm{s}} = -J \sum_n (\boldsymbol{S}_n \boldsymbol{S}_{n+1} + \delta Q_n^{zz}) \tag{61}$$

and, in the vicinity of the ground state, we have

$$H_s = -JN - J\sum_n (\delta Q_n^{zz} - q_n^2) \tag{62}$$

the Hamiltonian containing a pair product of spin operators.

Let us consider again the ground state of the easy-axis type. We have (see equation (22))

$$S^+ = \sqrt{2}\frac{\bar{\psi}_1 + \psi_1\bar{\psi}_2}{1 + |\psi_1|^2 + |\psi_2|^2} = 0$$

if

1. $\psi_1 = \psi_2 = 0$ and $S^z = 1, q^2 = 0$; this state is similar to the $\mathbb{CP}^1$ case.
2. $\psi_1 = 0$, ψ_2 is arbitrary:

$$S^z = \frac{|\psi_2|^2 - 1}{|\psi_2|^2 + 1} < 1 \qquad q^2 = 4\frac{|\psi_2|^2}{1 + |\psi_2|^2}\,.$$

3. By expressing $\psi_n = |\psi_n|e^{i\phi_n}$ one has $e^{-i\phi_1} + |\psi_2|e^{i(\phi_1 - \phi_2)} = 0$ or ψ_1 is arbitrary and $\psi_2 = e^{i(2\phi_1 - \pi)}$.

It is easy to check that the first solution has the minimal energy

$$E = -JN(1 + \delta) \qquad \delta > 0.$$

The same result can be expressed in terms of the angle parametrization (14), where

$$H_s = -\sum_n \left[\cos^2 2g + \delta\left(1 - \tfrac{1}{2}\sin^2\theta(1 - \cos 2\gamma \sin 2g)\right)\right] \tag{63}$$

and if $\delta > 0$ we have $\sin\theta = 0$ or $\theta = 0$ and $g = 0$, i.e. the known $\mathbb{CP}^1$ easy-axis vacuum.

Longer calculations show that for $\delta < 0$, we arrive at the easy-plane ground state: $S^z = 0$ but now there is a spin value reduction:

$$S^2 = 1 - \tfrac{1}{16}\delta^2 \qquad |\delta| < 4 \tag{64}$$

and we have the minimal energy

$$E = -JN(1 - \tfrac{1}{2}\delta + \tfrac{1}{16}\delta^2) \tag{65}$$

at

$$\sin^2 g = \tfrac{1}{16}\delta^2 \qquad \theta = 0. \tag{66}$$

The same expressions in the complex parametrization are readily given by

$$|\psi_2|^2 = 1 \tag{67}$$

$$|\psi_1|^2 = 2\left(\frac{4-\delta}{4+\delta}\right) \tag{68}$$

$$\phi_2 = 2\phi_1\,. \tag{69}$$

For the $\mathbb{CP}^3$ case, it is easy to show that ground states for model (1) are the same in terms of the classical spin vectors and $q^2 = g = 0$. Moreover, here again $\partial_t S^2 = 0$. In this case model (2) requires further study.

5. Classical lattice equations of motion

To obtain the classical equations of motion, following [2, 6] we consider the transition amplitude from the CS $|\psi\rangle$ at time t to the state $|\psi_1\rangle$ at instant t_1:

$$P(\psi_1, t_1|\psi, t) = \langle\psi_1| \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_1 - t)\right\}|\psi\rangle\,.$$

Dividing up the interval $t_1 - t$ into n equal subintervals $\epsilon = \frac{1}{n}(t_1 - t)$ and passing to the limit $n \to \infty$, we have

$$P(\psi_1, t_1|\psi, t) = \lim_{n\to\infty}\langle\psi_1|\left(1 - \frac{\mathrm{i}}{\hbar}\hat{H}\epsilon_1\right)\left(1 - \frac{\mathrm{i}}{\hbar}\hat{H}\epsilon_2\right)\cdots\left(1 - \frac{\mathrm{i}}{\hbar}\hat{H}\epsilon_n\right)|\psi_n\rangle\,.$$

By using the fact that the GCS obey the relation

$$\int \mathrm{d}\mu_i(\psi)\ |\psi\rangle\langle\psi| = I$$

we then obtain

$$P = \lim_{n\to\infty}\int\cdots\int\left[\prod_{k=1}^{n-1}\mathrm{d}\mu(\psi_k)\right]\prod_{k=1}^{n}\langle\psi_k|\left(1 - \frac{\mathrm{i}}{\hbar}\hat{H}\epsilon\right)|\psi_{k-1}\rangle$$

$$= \lim_{n\to\infty}\int\cdots\int\left[\prod_{k=1}^{n-1}\mathrm{d}\mu(\psi_k)\right]\prod_{k=1}^{n}\langle\psi_k|\psi_{k-1}\rangle\left(1 - \frac{\mathrm{i}\epsilon}{\hbar}\frac{\langle\psi_k|\hat{H}|\psi_{k-1}\rangle}{\langle\psi_k|\psi_{k-1}\rangle}\right)$$

with $\psi_0 = \psi$ and $\psi_n = \psi_1$. In the limit $\epsilon \to 0$ we have

$$1 - \frac{\mathrm{i}\epsilon}{\hbar}\frac{\langle\psi_k|\hat{H}|\psi_{k-1}\rangle}{\langle\psi_k|\psi_{k-1}\rangle} = \exp\left\{-\frac{\mathrm{i}\epsilon}{\hbar}\frac{\langle\psi_k|\hat{H}|\psi_{k-1}\rangle}{\langle\psi_k|\psi_{k-1}\rangle}\right\}\,.$$

Now since $\epsilon \ll 1$, we have $\psi_{k-1} = \psi_k - \Delta\psi_k$ and then

$$\langle\psi_k|\psi_{k-1}\rangle = 1 - \partial_{\psi'}\langle\psi_k|\psi'\rangle|_{\psi'=\psi_k}\Delta\psi_k - \partial_{\bar{\psi}'}\langle\psi_k|\psi'\rangle|_{\psi'=\psi_k}\Delta\bar{\psi}_k$$

$$= 1 + \frac{1}{2}\frac{\psi_{1k}\Delta\bar{\psi}_{1k} + \psi_{2k}\Delta\bar{\psi}_{2k} - \mathrm{CC}}{1 + |\psi_1|^2 + |\psi_2|^2} + \mathrm{O}[(\Delta\psi_i)^2]$$

or

$$\langle\psi_k|\psi_{k-1}\rangle \cong \exp\left\{\frac{1}{2}\frac{\psi_{1k}\Delta\bar{\psi}_{1k} + \psi_{2k}\Delta\bar{\psi}_{2k} - \mathrm{CC}}{1 + |\psi_1|^2 + |\psi_2|^2}\right\}$$

then

$$\prod_{k=1}^{n}\langle\psi_k|\psi_{k-1}\rangle = \exp\left\{\frac{1}{2}\sum\frac{\psi_{1k}\Delta\bar{\psi}_{1k} + \psi_{2k}\Delta\bar{\psi}_{2k} - \mathrm{CC}}{1 + |\psi_1|^2 + |\psi_2|^2}\right\}$$

$$= \exp\left\{\frac{1}{2}\int \mathrm{d}t\ \frac{\psi_{1k}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\psi}_{1k} + \psi_{2k}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\psi}_{2k} - \mathrm{CC}}{1 + |\psi_1|^2 + |\psi_2|^2}\right\}\,.$$

Finally, combining all the expressions, we obtain

$$P(\psi_1, t_1|\psi, t) = \int \mathrm{D}\mu(\psi)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'}\mathrm{d}\tau\ \mathcal{L}(\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2)\right\}$$

where

$$\mathcal{L} = \frac{\frac{1}{2}\mathrm{i}\hbar}{1+|\psi_1|^2+|\psi_2|^2}\sum_{l=1}^{2}\left(\psi_l \frac{\mathrm{d}}{\mathrm{d}t}\bar{\psi}_l - \mathrm{CC}\right) - \mathcal{H} \tag{70}$$

with

$$\mathcal{H} = \langle\psi_k|\hat{H}|\psi_k\rangle \tag{71}$$

at a lattice site.

The total classical lattice Lagrangian is the sum over all sites

$$L = \sum_{n=1}^{N} L_n \tag{72}$$

where $L_n = \mathcal{L}$ at site n. By varying (72) with respect to $\bar{\psi}_1$ and $\bar{\psi}_2$, one has at a site

$$\frac{\mathrm{i}}{1+|\psi_1|^2+|\psi_2|^2}\{\dot{\psi}_1(1+|\psi_2|^2) - \psi_1\bar{\psi}_2\dot{\psi}_2\} = \frac{\partial\mathcal{H}}{\partial\bar{\psi}_1}$$

$$\frac{\mathrm{i}}{1+|\psi_1|^2+|\psi_2|^2}\{\dot{\psi}_2(1+|\psi_1|^2) - \psi_2\bar{\psi}_1\dot{\psi}_1\} = \frac{\partial\mathcal{H}}{\partial\bar{\psi}_2}$$

or for the lattice

$$\mathrm{i}\dot{\psi}_{1n} = \left(1+|\psi_{1n}|^2+|\psi_{2n}|^2\right)\left((1+|\psi_{1n}|^2)\frac{\partial H}{\partial\bar{\psi}_{1n}} + \psi_{1n}\bar{\psi}_{2n}\frac{\partial H}{\partial\bar{\psi}_{2n}}\right) \qquad (1 \rightleftharpoons 2) \tag{73}$$

where $H = \sum_n H_n$ is the classical lattice Hamiltonian.

In the simplest case of weak exchange anisotropy, we have ($\mathbb{CP}^1$ case)

$$\frac{1}{j^2 J}H_{em} = -2\frac{\bar{\psi}_n\psi_{n+1} + \mathrm{CC}}{(1+|\psi_n|^2)(1+|\psi_{n+1}|^2)} - (1+\delta)\frac{(1-|\psi_n|^2)(1-|\psi_{n+1}|^2)}{(1+|\psi_n|^2)(1+|\psi_{n+1}|^2)} \tag{74}$$

and the equations of motion (see [2])

$$\mathrm{i}\dot{\psi}_n = (1+|\psi_n|^2)^2\frac{\partial H}{\partial\bar{\psi}_n}$$

i.e.

$$\frac{1}{2}\mathrm{i}\dot{\psi}_n = \frac{\psi_{n-1}}{1+|\psi_{n-1}|^2} + \frac{\psi_{n+1}}{1+|\psi_{n+1}|^2} - \psi_n^2\left(\frac{\bar{\psi}_{n-1}}{1+|\psi_{n-1}|^2} + \frac{\bar{\psi}_{n+1}}{1+|\psi_{n+1}|^2}\right)$$

$$-2(1+\delta)\psi_n\frac{1-|\psi_{n-1}|^2|\psi_{n+1}|^2}{(1+|\psi_{n-1}|^2)(1+|\psi_{n+1}|^2)} \tag{75}$$

which, in the small amplitude region, is

$$\tfrac{1}{2}\mathrm{i}\dot{\psi}_n = \psi_{n-1} + \psi_{n+1} - 2(1+\delta)\psi_n - (\psi_{n-1}|\psi_{n-1}|^2 + \psi_{n+1}|\psi_{n+1}|^2)$$

$$-\psi_n^2(\bar{\psi}_{n-1} + \bar{\psi}_{n+1}) + \psi_n(|\psi_{n-1}|^2 + \psi_{n+1}|^2) \tag{76}$$

the equation of the so-called ϕ_4-theory. It is easy to verify that equation (76) possesses two ground states solutions:

$$\psi_n = \psi_{n\pm1}$$

then

$$-2\delta\psi_n\frac{1-|\psi_n|^4}{(1+|\psi_n|^2)^2} = 0$$

and

$$\psi_n = 0 \qquad \text{(easy-axis vacuum state)} \tag{77}$$

$$|\psi_n|^2 = 1 \qquad \text{(easy-plane vacuum state).} \tag{78}$$

Equation (71) has only the easy-axis vacuum.

For $s = \frac{3}{2}$, instead of (73) one has

$$\mathrm{i}\dot{\psi}_{1n} = (1 + |\psi_{1n}|^2 + |\psi_{2n}|^2 + |\psi_{3n}|^2)\left\{(1 + |\psi_{1n}|^2)\frac{\partial H}{\partial \bar{\psi}_{1n}} + \psi_{1n}\bar{\psi}_{2n}\frac{\partial H}{\partial \bar{\psi}_{2n}} + \psi_{1n}\bar{\psi}_{3n}\frac{\partial H}{\partial \bar{\psi}_{3n}}\right\} \qquad (1 \rightleftharpoons 2) \quad (1 \rightleftharpoons 3) \tag{79}$$

or for arbitrary s (and $\mathbb{CP}^{2s}$ GCS):

$$\mathrm{i}\dot{\psi}_{jn} = \left(1 + \sum_{k=1}^{2s} |\psi_{kn}|^2\right)\left\{(1 + |\psi_{jn}|^2)\frac{\partial H}{\partial \bar{\psi}_{jn}} + \psi_{jn}\sum_{k=1}^{2s}\bar{\psi}_{jk}\frac{\partial H}{\partial \bar{\psi}_{jk}}\alpha_{1k}\right\} \tag{80}$$

with

$$\alpha_{1k} = \left\{\begin{array}{ll} 0 & \text{if } \; k = 1 \\ 1 & \text{if } \; k \neq 1 \end{array}\right\}.$$

6. Classical continuum equations of motion

To obtain equations of motion in the continuum approximation, we apply the conventional procedure of expanding lattice functions, $\psi_{n\pm1}$, in the Taylor series up to the second order derivatives (the first non-vanishing terms) supposing $\lambda a_0 \ll 1$:

$$\psi_{n\pm1} = \psi(x) \pm a_0\psi'(x) + \tfrac{1}{2}a_0^2\psi''(x) + \mathrm{O}(a_0^3\psi''')$$

where $x = a_0 n$.

Consider first for the sake of simplicity the $\mathbb{S}^2$ and $\mathbb{S}^{1,1}$ cases. Then the equations of motion become

$$\mathrm{i}\dot{\psi} + \triangle\psi - 2\frac{(\nabla\psi)^2\bar{\psi}}{1 + |\psi|^2} + \delta\frac{1 - |\psi|^2}{1 + |\psi|^2} = 0 \tag{81}$$

for the ferromagnetic case and

$$\mathrm{i}\dot{\zeta} + \triangle\zeta + 2\frac{(\nabla\zeta)^2\bar{\zeta}}{1 - |\zeta|^2} + \delta\frac{1 + |\zeta|^2}{1 - |\zeta|^2} = 0 \tag{82}$$

for the antiferromagnetic case, where $\Delta = \nabla^2$ is the Laplace operator. For one-space dimensional systems these equations, being the stereographic projections of compact and non-compact Landau–Lifshitz models defined respectively on the sphere $\mathbb{S}^2$ and the hyperboloid $\mathbb{S}^{1,1}$, are gauge equivalent to various integrable versions of the nonlinear Schrödinger equation (NSE) [9]. In the particular case of $\delta > 0$, equation (81) is equivalent to the cubic attraction type NSE and equation (82) to the repulsive type NSE. The latter, usually called the Ginzburg–Landau equation, describes superfluid phenomenology and gives the correct Bogolubov excitation spectrum. What is probably more intriguing is that the σ-model version of (82) even gives a correct quasiclassical description of the Bogolubov condensate, thereby pointing out at the intimate coupling of antiferromagnetism and superfluidity. Quite a detailed discussion of this equivalence, along with solutions of the equations can be found in [2, 9].

Naturally, the question arises as to what extent these continuum models can be related to the initial lattice ones. It is easy to show that at $D \geqslant 2$ such continuum systems lose stationary localized solutions. In what follows, our goal will be the $\mathbb{CP}^2$ model. In this regard, as we have already seen, the most interesting is model (2) with single-ion anisotropy. Equations of motion then assume the simplest form in terms of real functions $(\theta, \phi, \gamma, g)$ and $D = 1$:

$$
\begin{aligned}
\dot{\phi} &= \frac{\cos 2g}{\sin\theta}\theta_{xx} - \cos 2g \cos\theta\phi_x^2 - 4\frac{\sin 2g}{\sin\theta} g_x\theta_x + \delta\frac{\cos\theta}{\cos 2g}(4\sin 2g\cos 2\gamma - 1) \\
\dot{\theta} &= -\cos 2g(\sin\theta\phi_{xx} + 2\cos\theta\phi_x\theta_x) + 4\sin 2g\sin\theta g_x\theta_x \\
&\qquad - \frac{\delta}{2}\tan 2g\sin 2\theta\sin 2\gamma \\
\dot{\gamma} &= 2\sin 2g(g_{xx} + 2\cot\theta g_x\theta_x) - \cos 2g(\cot\theta\theta_{xx} - 4g_x^2 - \theta_x^2 - \phi_x^2) - 2\cos 2g \\
&\qquad + \delta\left[\frac{\cos^2\theta}{\cos 2g}(1 - \sin 2g\cos 2\gamma) + \frac{1}{2}\cot\sin^2\theta\cos 2\gamma\right] \\
\dot{g} &= \frac{\delta}{2}\sin 2\gamma\sin^2\theta\,.
\end{aligned}
\tag{83}
$$

In order to make the models comparable we give the equations for the exchange anisotropy model (1):

$$
\begin{aligned}
\dot{\phi} &= \frac{\cos 2g}{\sin\theta}\theta_{xx} - \cos 2g\cos\theta\phi_x^2 - 4\frac{\sin 2g}{\sin\theta}g_x\theta_x - 2\delta\cos 2g\cos\theta \\
\dot{\theta} &= -\cos 2g(\sin\theta\phi_{xx} + 2\cos\theta\phi_x\theta_x) + 4\sin 2g\sin\theta g_x\theta_x \\
\dot{\gamma} &= 2\sin 2g(g_{xx} + 2\cot\theta g_x\theta_x) - \cos 2g(\cot\theta\theta_{xx} - 4g_x^2 - \theta_x^2 - \phi_x^2) - 2\cos 2g \\
\dot{g} &= 0\,.
\end{aligned}
\tag{84}
$$

From equations (83) and (84), one can easily infer conclusions (i) and (ii) of section 7 and that the system (84) reduces to the Landau–Lifshitz equations when $g = 0$.

7. Conclusion

Summarizing the results, we can infer the following:

(i) Spherically symmetric systems and systems with exchange anisotropy, regardless of its magnitude and sign, can be treated via $\mathbb{CP}^1$ GCS because the Hamiltonian in this case does not contain correlators, and S^2 and q^2 are conserved separately, $\partial_t S^2 = \partial_t q^2 = 0$ (where q is a cyclic coordinate). Since in both ground states (or in the domain regions) $q^2 = 0$, the same thing occurs in the transition (intermediate) region, the domain wall. From the physical point of view, these nonlinear systems behave, in a sense, as linear for they do not excite higher 'harmonics', in our case, higher moments. The Landau–Lifshitz phenomenology should work well for these and if, nevertheless, they display certain spin reduction, it should be attributed, in the scope of our consideration, to stochastic (chaotic) processes. This is one of the goals in investigating the underlying lattice models.

(ii) Systems with single-ion anisotropy, regardless of the anisotropy coefficient and its sign, should be treated via $\mathbb{CP}^2$ GCS for $s = 1$ and higher $\mathbb{CP}^{2s}$ for higher spins. Here pair products of the spin operators at the same site are in the Hamiltonians and therefore $\partial_t S^2 \neq 0$. Ground states can also depend on q (see equations (66) and (68)) so spin reduction can occur even in the ground states and, in principle, may be attributed to both mechanisms: excitations of higher moments and stochastic processes.

(iii) The same conclusions have to apply for antiferromagnets and $SU(1,1)/U(1)$ GCS.

(iv) For systems with sufficiently large exchange anisotropy, it is necessary to develop techniques based on the q-deformed algebras, in the sense that the algebra deformation is related to anisotropy rate.

Along with impressive results, obtained for the models considered within the scope of the continuum approximation, especially concerning their dynamical properties, there is still, in fact, nearly nothing serious about the dynamics of the corresponding lattice models. Therefore, the study of the dynamical behaviour of such models is of a great interest from both the theoretical point of view and that of physical applications. This study should consist of analytical research as well as computer modelling.

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References

[1] Izergin A and Korepin V 1982 *Quantum Method of Inverse Problem (Particles and Nuclei 13)* (Moscow) pp 501–41
[2] Makhankov V 1990 *Soliton Phenomenology* (Dordrecht: Kluwer)
[3] Perelomov A 1987 *Generalized Coherent States and Applications* (Moscow: Nauka)
[4] Aragone C, Chalbaud E and Salamó S 1963 *J. Math. Phys.* **17** 176
Lieb E H 1973 *Commun. Math. Phys.* **31** 327
Janssen D 1977 *Sov. J. Nucl. Phys.* **25** (4) 479
[5] Mead L and Papanicolaou N 1982 *Phys. Rev.* B **26** 1416–29
[6] Kuratsuji H and Suzuki T 1980 *J. Math. Phys.* **21** 472–80
Kuratsuji H and Susuki T 1983 *Progr. Theor. Phys. Suppl.* **74&75** 209
[7] Makhankov V, Myrzakulov R and Makhankov A 1987 *Phys. Scr.* **35** 233
Makhankov V and Makhankov A 1988 *Phys. Stat. Solidi* b **145** 669
[8] Ostrovskii V 1986 *Zh. Exp. Teor. Fiz.* **91** 1690–1701
[9] Makhankov V and Pashaev O 1993 Integrable pseudospin models in condensed matter *Sov. Sci. Rev. G.–Math. Phys.* vol 9 (London: Harwood) pp 1-145